Percolation in a gradient: conductivity properties

1990 J. Phys. A: Math. Gen. 23 L1253

(http://iopscience.iop.org/0305-4470/23/23/012)

J. Phys. A: Math. Gen. 23 (1990) L1253-L1258. Printed in the UK

LETTER TO THE EDITOR

Percolation in a gradient: conductivity properties

Stéphane Roux†§, Alex Hansen‡ and Einar L Hinrichsen‡

† Centre d'Enseignement et de Recherche en Analyse des Matériaux, Ecole Nationale des Ponts et Chaussées, Central IV, 1 Av. Montaigne, F-93167 Noisy-le-Grand Cédex, France
‡ Fysisk Institutt, Universitetet i Oslo, Postboks 1048, Blindern, N-0316 Oslo 3, Norway

Received 2 October 1990

Abstract. We investigate the conductivity properties of a randomly diluted medium where the fraction of present bonds varies along the mean voltage drop, and reaches the percolation threshold. We obtain the scaling of the conductivity as a function of the system size for any concentration profile. We show that a transient scaling regime also appears for small system sizes and a rapidly varying concentration of bonds close to the threshold. Finally the distribution of local currents is also investigated.

Much effort has been spent trying to characterize the transport properties of systems close to the percolation threshold. In particular, for random resistor networks, where the fraction of conducting bonds, p, is uniform, the scaling of the conductivity as a function of p and of the system size is well known [1].

Inhomogeneous dilution, the so-called 'percolation in a gradient' problem, has been considered mainly for its geometrical properties [2]. In particular, it has been shown that it provides a very efficient tool for obtaining an accurate determination of the percolation threshold in two dimensions [3].

The aim of this letter is to show that the scaling of transport properties of inhomogeneous systems, where p depends on the position, can be readily analysed using the results derived for homogeneous systems. We will mainly address two points: first, the scaling of the conductivity in a lattice where the concentration of conducting bonds varies along the mean voltage drop and, second, the distribution of local currents in such a system.

Let $p(y)$ be the concentration of present bonds in a strip of width w, at a distance y from one border. We consider specifically the following profiles:

$$p(y) = p_c + (1 - p_c)\left(\frac{y}{w}\right)^{\zeta}. \tag{1}$$

We ask the following question: when the width of the strip goes to infinity, what is the scaling of the transverse conductivity of the strip with the width w?

We will answer this question in two steps, and show numerical simulation which confirms our theoretical expectation. First we will show that if ζ is large enough, the conductivity of the strip will be controlled by that of the most poorly conducting part. However, due to the finite width of the strip and of the non-monotonous profile $p(y)$,

§ Also at Laboratoire de Physique et Mécaniques des Milieux Hétérogènes, URA CNRS 857, Ecole Supérieure de Physique et Chimie Industrielles, 10 rue Vauquelin, F-75231 Paris Cédex 05, France.

0305-4470/90/231253+06$03.50

the effective lower value of p that will determine the conductivity is not p_c but will evolve with w. Once this effective cutoff is determined, the computation of the conductivity is straightforward. We will discuss some transient scaling regimes for low values of ζ and small widths.

Using the fact that the correlation length, ξ, diverges close to the threshold as $\xi \sim |p-p_c|^{-\nu}$, we propose as a criterion for the effective cutoff, the value of $y = y^*$ such that the correlation length at this point $\xi(y^*)$ matches y^* exactly. In two dimensions, the correlation length exponent $\nu = \frac{4}{3}$. Up to a constant prefactor, this statement can be written as

$$(p(y^*) - p_c)^{-\nu} = y^* \tag{2}$$

or

$$y^*(w) \sim w^{\zeta\nu/(1+\zeta\nu)} \tag{3}$$

where $y^*(w)$ is the distance where the effective cutoff in p is reached. This kind of effective cutoff is used classically in the percolation problem in a gradient or, more generally, in many instances of critical phenomena with a spatial or temporal variation of the control parameter [3, 4].

For small strip width, and small values of ζ, we may, however, have to consider an alternative cutoff such that the distance to the border where $p = 1$ is more restrictive than the distance to the $p = p_c$ border. In such a case, we introduce a second length scale, y^{**}, such that $\xi(y^{**}) = w - y^{**}$. Assuming $y^{**} \ll w$, we obtain

$$y^{**}(w) \sim w^{(\zeta\nu-1)/\zeta\nu}. \tag{4}$$

Note that y^{**}/w decreases with w and thus the condition $y^{**} \ll w$ is naturally fulfilled for a large enough width. Since $(\zeta\nu - 1)/\zeta\nu < \zeta\nu/(1+\zeta\nu)$, for large system sizes, the first cutoff, y^*, is more stringent than the second one, y^{**}. Therefore, the latter cutoff can only appear in a transient regime. However, we will see numerically that some cases have to be analysed using this cut-off.

We can view the strip as a continuous conducting material with varying conductivity along the mean voltage drop. In this case, the resistivity $\rho(w)$ is simply given by the sum:

$$\rho(w) = \frac{1}{w}\int_0^w r(y)\,\mathrm{d}y \tag{5}$$

where $r(y)$ is the resistivity of the strip at a distance y from the border. We may now use the divergence of $r(p)$ for a homogeneous system close to the threshold, characterized by an exponent t ($t \approx 1.300$ in two dimensions [5])

$$r(p) \sim (p - p_c)^{-t}. \tag{6}$$

Combining this with (5), we obtain

$$\rho(w) \sim \frac{1}{w}\int_0^w \left(\frac{y}{w}\right)^{-\zeta t} \mathrm{d}y. \tag{7}$$

This integral is not convergent when the exponent ζ is larger than $1/t \approx 0.77$. This means, physically, that in this case the resistivity of the strip will be controlled by the lower cutoff introduced previously. In contrast, if $\zeta < 1/t$ then ρ is finite (constant), and does not scale with the width of the strip.

Let us now concentrate on the case $\zeta > 1/t$. As mentioned previously, we can simply change the lower bound 0 in the integral (7), into y^* from (3). Integration gives the final answer

$$\rho(w) \sim w^{x(\zeta)} \tag{8}$$

where

$$x(\zeta) = \frac{(\zeta t - 1)}{(\zeta \nu + 1)}. \tag{9}$$

Let us return for a moment to the effective cutoff determined by the discussion leading to (4). When the exponent ζ is small (although larger than $1/t$ for the scaling to hold) it might be that, when the width of the strip is small, the second cutoff y^{**} appears to be the more restrictive. In such a case, the scaling of ρ gives

$$\rho(w) \sim w^{x_s(\zeta)} \tag{10}$$

where

$$x_s(\zeta) = \frac{(\zeta t - 1)}{\zeta \nu}. \tag{11}$$

We have performed numerical simulation of the problem using a transfer-matrix method. The length of the strip used was 10^5 in all cases. The width w ranges from 2–20 (30 in some cases). We tested different exponents ζ: 0.5, 1, 2, and 3 and obtained a good agreement with the results presented above.

For $\zeta = 0.5$ (figure 1) we clearly see a curvature such that the tangent exponent measured, 0.09, is certainly an upper bound. Thus we have $x_s(0.5) < 0.09$, in agreement with the expected result 0 (since $\zeta < 1/t$).

For $\zeta = 2$ (figure 2) and $\zeta = 3$, respectively, we measured a value of x equal to 0.43 and 0.55, whereas the expected values are 0.437 and 0.58.

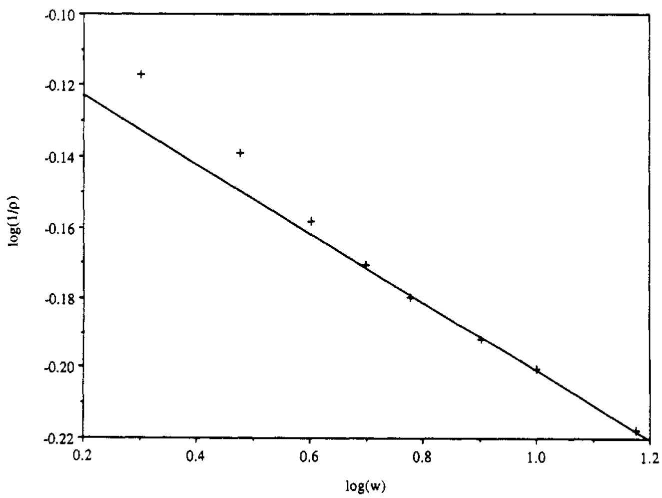

Figure 1. Log–log plot of the conductivity $1/\rho$ as a function of the strip width w for the profile given by equation (1) with $\zeta = 0.5$. The slope of the straight line is 0.09.

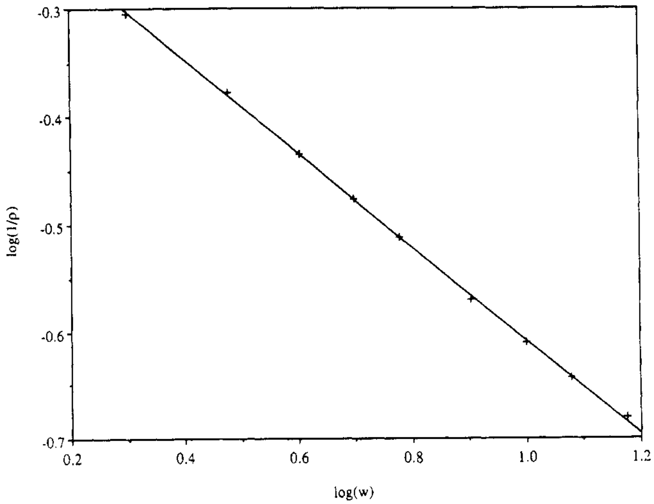

Figure 2. Log-log plot of the conductivity $1/\rho$ as a function of the strip width w for the profile given by equation (1) with $\zeta = 2$. The slope of the straight line is 0.43.

For $\zeta = 1$ (figure 3) we see for $2 < w < 30$ an apparent exponent 0.21, which can be compared to the transient regime exponent $x_s(1) = 0.2245$. However, a clear curvature can be seen which suggests that the tangent exponent decreases with the system size. The expected asymptotic value is 0.128.

Apparently, the transient regime cannot be seen for large values of ζ (2 or more). Most probably it should occur for very small widths and is thus hidden by corrections to scaling effects.

In plain homogeneous systems, at percolation threshold, the distribution of local currents has been shown to be multifractal [6, 7]. In simple terms, this property states that the log-log histogram of the current distribution, rescaled by the logarithm of the system size, is size independent. Let us define the rescaled current $\alpha = -\log(i)/\log(L)$

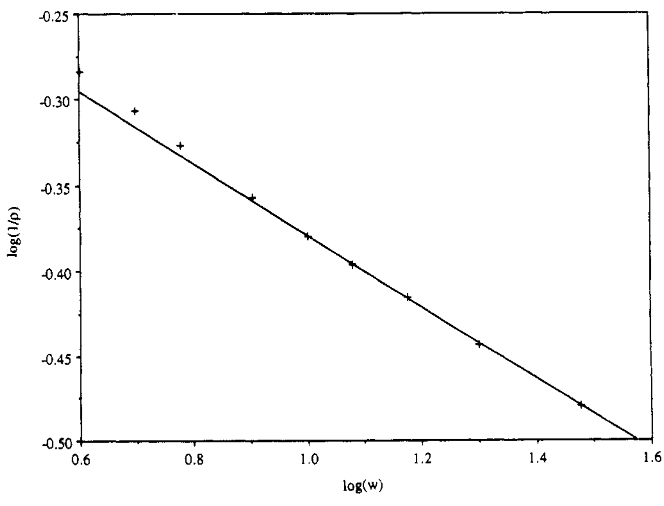

Figure 3. Log-log plot of the conductivity $1/\rho$ as a function of the strip width w for the profile given by equation (1) with $\zeta = 1$. The slope of the straight line is 0.21.

where i is the local current, and L the system size. The rescaled density $f = \log(n(i))/\log(L)$, where $n(i)$ is the logarithmically-binned current distribution. $f(\alpha)$ is a universal curve independent of the system size.

In the gradient problem, we can also consider the current distribution in these rescaled variables. Distributions obtained for different system sizes collapse onto a single curve, as shown in figure 4 for the case of a linear profile ($\zeta = 1$). The numerical data were generated using square lattices of size 10×10, 15×15 and 20×20, (averaged over 100 lattices for the largest size). The current distribution was solved using a conjugate gradient algorithm with a precision of 10^{-10}.

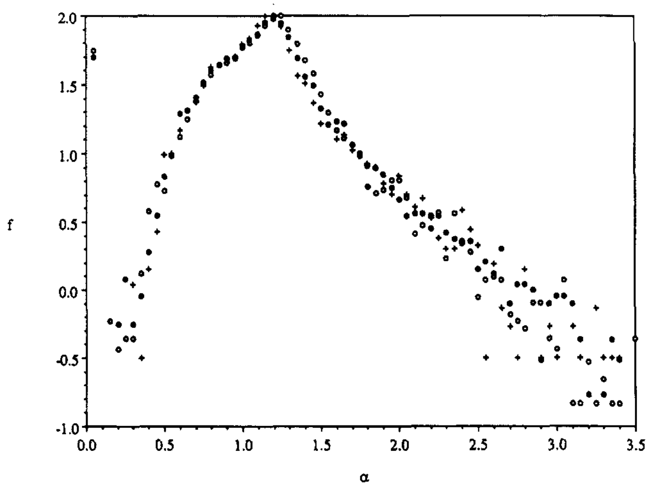

Figure 4. Current distribution in terms of rescaled variables f and α defined in the text. A good data collapse is obtained for three system sizes 10 (○), 15 (●) and 20 (+), with a linear variation of p (equation (1) with $\zeta = 1$).

The shape of the $f(\alpha)$ function in this case can be analysed using exactly the same formalism as developed in [8]. In [8], a temporal variation of p was considered, rather than a spatial one. However, the conclusion is identical. The $f(\alpha)$ in the case of a non-homogeneous system can be obtained by taking the convex envelope of the multifractal spectrum at the percolation threshold, plus the point of coordinate $\alpha = 1$, and $f = 2 + 1/\zeta\nu$. Thus we expect to see a 'wedge shape' for α close to 1 (i.e. two power-law behaviours, $n(i) \propto i^{\phi}$) which ends for large and small values of α with the multifractal spectrum at the effective cutoff value of p. This is confirmed by the shape of the histogram shown on figure 4. In particular the exponent ϕ for α less than 1, is consistent with the expected value 1.6 reported in [8]. The fact that this exponent is less than 2 means that the second moment of the current distribution (the conductivity) will be controlled by the values of α that come from the curved part of the histogram, i.e. the values relative to the multifractal spectrum at the effective cutoff for p close to p_c. When ζ is less than $1/t$ then the exponent ϕ is larger than 2; thus the conductivity gets its dominant contribution from the top of the histogram, i.e. from bonds in the bulk of the medium. This conclusion is consistent with our previous expectation: that the conductivity did not scale with the system size for ζ less than $1/t$.

Let us conclude by noting that despite the fact that we concentrated our study on simple profiles, (1), the results obtained are much more general. Since the medium can be seen as a collection of 'slices' of different conductivities placed in series, the ordering of the local conductivities is not relevant. Thus $p(y)$ can be seen as the distribution of the local conductivities, regardless of their precise location in space. Moreover, the behaviour of $p(y)$ close to p_c (the equivalent of our parameter ζ) is the only factor that will determine the scaling properties of the conductivity, and of the current distribution.

We acknowledge useful discussion with A Aharony, J Feder, and T Jøssang. AH and ELH are supported by the German-Norwegian Research Cooperation.

References

[1] Aharony A 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum) p 289
[2] Sapoval B, Rosso M and Gouyet J F 1985 *J. Physique Lett.* **46** L149
[3] Ziff R and Sapoval B 1986 *J. Phys. A: Math. Gen.* **19** 1169
[4] Diehl H W 1986 *Phase Transitions and Critical Phenomena* vol 10, ed C Domb and J L Lebowitz (New York: Academic)
[5] Normand J M, Herrmann H J and Hajjar M 1988 *J. Stat. Phys.* **52** 441
[6] Rammal R, Tannous C, Breton P and Tremblay A M S 1985 *Phys. Rev. Lett.* **54** 1718
[7] de Arcangelis L, Redner S and Coniglio A 1985 *Phys. Rev.* B **31** 4725; 1986 *Phys. Rev.* B **34** 4656
[8] Roux S and Hansen A 1989 *Europhys. Lett.* **8** 729; *Disorder and Fracture (Proc. Cargese NATO ASI)* ed J C Charmet, E Guyon and S Roux (New York: Plenum) to appear